Quantization of waves in dispersive media with application to nonlinear interactions of plasmons

1972 J. Phys. A: Gen. Phys. 5 1578

(http://iopscience.iop.org/0022-3689/5/11/006)

J. Phys. A: Gen. Phys., Vol. 5, November 1972. Printed in Great Britain. © 1972

Quantization of waves in dispersive media with application to nonlinear interactions of plasmons

J ASKNE

Research Laboratory of Electronics, Chalmers University of Technology, Gothenburg, Sweden

MS received 2 May 1972

Abstract. Waves in a general dispersive medium represented by a quadratic or bilinear Lagrangian are quantized. A phenomenological method is used leading to medium excitations in terms of quasiparticles (bosons) which are photon-like, phonon-like etc. The possibility for negative energy excitations is considered and interactions between different wave types studied. The simplicity of the analysis is demonstrated by an analysis of nonlinear interactions in a multistream medium of plasmons.

1. Introduction

The normal procedure to quantize wave motion in a material medium, is to consider the different waves or oscillations in the material medium separately and to diagonalize the total Hamiltonian, see for example Hopfield (1958), Fukai and Harris (1971). The other possibility is to introduce the phenomenological constants, such as ϵ and μ for electromagnetic waves, which in the dispersive medium are functions of frequency and wavelength. Furthermore we then consider field quantities which are averaged over volumes containing a large number of particles. Such a method should be useful particularly when we consider interaction between two complicated systems each described by their phenomenological constants.

The phenomenological method has been applied in the electromagnetic, nondispersive case by Jauch and Watson (1948) and Kong (1970), and applied to Čerenkov radiation, while Alekseev and Nikitin (1966) studied the electromagnetic field in a nonmagnetic dispersive medium with application to the radiation from an impurity atom. This method has recently been of value in studies of linear and nonlinear interactions in plasmas (Harris 1970). Even though for classical plasmas one may finally set Planck's constant equal to zero the use of quantum methods turns out to be quite useful. Applications of quantized wave motion are also given by Louisell *et al* (1961) and Musha (1964).

In this article a general dispersive medium represented by a quadratic or bilinear Lagrangian of general form is assumed and the solutions are expanded in wave modes. This leads us to a description of medium excitations in terms of quasiparticles of boson type (photons, phonons, plasmons, magnons etc). Due to interaction between different wave types we may rather speak of photon-like, phonon-like, etc excitations. Particularly the quantization of negative energy excitations will be considered. As we will allow for negative frequencies we will rather speak of negative action waves (action = energy divided by frequency). Although negative energy waves have been quantized in some special cases (cf Jauch and Watson 1948, Harris 1970, Musha 1964) a general theory does not seem to have been given. It is shown to be convenient to introduce creation and annihilation operators with different commutation rules for positive and negative action waves, resulting in a compact notation for an interaction analysis. As an illustration of the method wave motion in a multistream medium is considered as well as the weak nonlinear three wave interaction.

It should be remarked that instead of starting from the Lagrangian the present method will allow us to start directly from the differential equations (see Appendix).

2. Description of a dispersive medium

A classical system is quantized via the Lagrangian and Hamiltonian formalism. In the case of a disturbed system the Lagrangian density L is a function of the generalized coordinates q_i and the first order time and space derivatives of q_i, that is

$$L\left(q_i, \frac{\partial q_i}{\partial t}, \frac{\partial q_i}{\partial z}\right). \tag{1}$$

When the Lagrangian is known we can obtain all information about the system by means of the principle of least action

$$\delta \iint L \, \mathrm{d}z \, \mathrm{d}t = 0 \tag{2}$$

which yields the differential equations

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial q_i/\partial t)} + \frac{\partial}{\partial z}\frac{\partial L}{\partial(\partial q_i/\partial z)} - \frac{\partial L}{\partial q_i} = 0. \tag{3}$$

The characteristic feature in a description of a dispersive medium is that some (or all but one) of the coordinates are eliminated and the effective medium parameters are frequency and/or wavelength dependent. In order to do this we expand the coordinates in monochromatic wave solutions with periodic boundary conditions over unit length

$$q_i(t, z) = \tfrac{1}{2} \sum_{k=-\infty}^{+\infty} (Q_{i,k}\, \mathrm{e}^{\mathrm{i}kz} + Q_{i,k}^{+}\, \mathrm{e}^{-\mathrm{i}kz}). \tag{4}$$

The reality condition is

$$Q_{i,k} = Q_{i,-k}^{+} \tag{5}$$

Furthermore, we assume that the amplitude Q_k depends on time t through the factor $\exp(-\mathrm{i}\omega_k t)$, where $\omega_k = \omega(k)$ is a relation to be determined later. Up to that point ω_k and k are considered as independent variables. In the Appendix we have shown that if we assume a quadratic or bilinear Lagrangian of general form and eliminate all 'coordinates' but one, denoted $q(t, z)$ we obtain after integration over unit length for $\omega_k = -\omega_{-k}$ and ω_k real

$$\int L \, \mathrm{d}z \equiv \tfrac{1}{4} \sum_k \mathscr{L}_k(\omega_k, k)(Q_k Q_k^{+} + Q_k^{+} Q_k). \tag{6}$$

The corresponding expression for the momentum

$$p(t, z) = \tfrac{1}{2} \sum_k (P_k \, \mathrm{e}^{\mathrm{i}kz} + P_k^+ \, \mathrm{e}^{-\mathrm{i}kz}) \tag{7}$$

is

$$P_k = -\frac{\mathrm{i}}{2} \frac{\partial \mathscr{L}_k}{\partial \omega_k} Q_k \qquad P_k^+ = \frac{\mathrm{i}}{2} \frac{\partial \mathscr{L}_k}{\partial \omega_k} Q_k^+ . \tag{8}$$

We then obtain for the Hamiltonian and wave momentum

$$\int H \, \mathrm{d}z = \int \left(p(t, z) \frac{\partial q(t, z)}{\partial t} - L \right) \mathrm{d}z = \frac{1}{4} \sum_k \left(\omega_k \frac{\partial \mathscr{L}_k}{\partial \omega_k} - \mathscr{L}_k \right) (Q_k Q_k^+ + Q_k^+ Q_k) \tag{9}$$

$$\int \Pi \, \mathrm{d}z = \int \left(-p(t, z) \frac{\partial q(t, z)}{\partial z} \right) \mathrm{d}z = \frac{1}{4} \sum_k k \frac{\partial \mathscr{L}_k}{\partial \omega_k} (Q_k Q_k^+ + Q_k^+ Q_k) \tag{10}$$

or

$$\int H \, \mathrm{d}z = \sum_k H(t, k) = \tfrac{1}{4} \sum_k \mathscr{H}_k(\omega_k, k)(Q_k Q_k^+ + Q_k^+ Q_k) \tag{11}$$

with

$$\mathscr{H}_k = \omega_k \frac{\partial \mathscr{L}_k}{\partial \omega_k} - \mathscr{L}_k . \tag{12}$$

(11) together with (8) yields the canonical equations

$$\begin{aligned} \frac{\partial H(t, k)}{\partial P_k} &= \dot{Q}_k^+ & \qquad \frac{\partial H(t, k)}{\partial P_k^+} &= \dot{Q}_k \\ \frac{\partial H(t, k)}{\partial Q_k^+} &= -\dot{P}_k & \qquad \frac{\partial H(t, k)}{\partial Q_k} &= -\dot{P}_k^+ \end{aligned} \tag{13}$$

if

$$\mathscr{L}_k(\omega_k, k) = 0. \tag{14}$$

(14) yields the dispersion relation $\omega_{k,j} = \omega_j(k)$, where j denotes the jth solution. We observe that (14) always yields two identical sets of solutions. The usual one is $\omega_{k,j} > 0$ for $k > 0$. In the preceding expressions a summation over both sets is implied.

3. Quantization

The system is now quantized by replacing the coordinate and momentum determined by the canonical equations with noncommuting operators. For the boson fields the commutation relations then are

$$[Q_{kj}^+, P_{k'j}] = \mathrm{i}\hbar \, \delta_{kk'} \qquad [Q_{kj}, P_{k'j}^+] = \mathrm{i}\hbar \, \delta_{kk'} \tag{15}$$

where a summation over the two identical ω_{kj} solutions is implied. It may be remarked that the same commutation rules are obtained if P_k is replaced by $\alpha_k P_k$ at the same time as Q_k is replaced by $(\alpha_k)^{-1} Q_k$, where α_k is a c number.

We observe that the sign of $\partial \mathscr{L}_k/\partial \omega_k$ may be negative and this is the case which is denoted negative action waves (action = energy divided by frequency). This is the same as negative energy waves when the frequency is positive. (For convenience we will include positive as well as negative frequency solutions in this paper.) It is well known (Sturrock 1960), that negative energy waves are obtained for example in systems with drifting charge carriers. In such systems we have a certain energy associated with the constant motion of the beam. This energy is respectively increased or decreased if a positive or negative energy wave is excited. We introduce

$$s_{k,j} = \mathrm{sgn}\left(\frac{\partial \mathscr{L}_k}{\partial \omega_k}\right)_{\omega_{k,j}} \tag{16}$$

and the creation and annihilation operators may be defined by

$$\begin{aligned} a_k &= \frac{s_k}{(\hbar|\partial \mathscr{L}_k/\partial \omega_k|)^{1/2}}\left(\frac{1}{2}\frac{\partial \mathscr{L}_k}{\partial \omega_k}Q_k + \mathrm{i}P_k\right) = \left(\frac{|\partial \mathscr{L}_k/\partial \omega_k|}{\hbar}\right)^{1/2} Q_k \\ a_k^+ &= \frac{s_k}{(\hbar|\partial \mathscr{L}_k/\partial \omega_k|)^{1/2}}\left(\frac{1}{2}\frac{\partial \mathscr{L}_k}{\partial \omega_k}Q_k^+ - \mathrm{i}P_k^+\right) = \left(\frac{|\partial \mathscr{L}_k/\partial \omega_k|}{\hbar}\right)^{1/2} Q_k^+ . \end{aligned} \tag{17}$$

The commutation relation is according to (15)

$$[a_{k,j}, a_{k',j'}^+] = s_{k,j}\,\delta_{k,k'}\,\delta_{j,j'}.$$

In the following expressions we only sum over one of the two identical $\omega_{k,j}$ solutions as remarked after (14)

$$q(t,z) = \sum_k \sum_j \left(\frac{\hbar}{|(\partial \mathscr{L}_k/\partial \omega_k)_{\omega_{k,j}}|}\right)^{1/2} (a_{k,j}\,\mathrm{e}^{\mathrm{i}kz} + a_{k,j}^+\,\mathrm{e}^{-\mathrm{i}kz}) \tag{18}$$

$$H(t) = \sum_k \sum_j \tfrac{1}{2} s_{k,j}\hbar\omega_{k,j}(a_{k,j}a_{k,j}^+ + a_{k,j}^+ a_{k,j}) \tag{19}$$

$$\Pi(t) = \sum_k \sum_j \tfrac{1}{2} s_{k,j}\hbar k(a_{k,j}a_{k,j}^+ + a_{k,j}^+ a_{k,j}). \tag{20}$$

The time dependence of a_k is given by Heisenberg's equation of motion

$$\mathrm{i}\hbar\frac{\partial a_{k,j}}{\partial t} = [a_{k,j}, H] = \hbar\omega_{k,j}a_{k,j} \tag{21}$$

that is, the time dependence of $a_{k,j}$ is independent of $s_{k,j}$, which is convenient in analysing linear and nonlinear interactions.

The roles of creation and annihilation operators are interchanged when s_k change sign as a creation of a negative quanta means an annihilation of energy. This is described by table 1.

Table 1.

$s_k = +1$	$s_k = -1$
$a_k\|n_k\rangle = \sqrt{n_k}\|n_k - 1\rangle$	$a_k^+\|n_k\rangle = \sqrt{n_k}\|n_k - 1\rangle$
$a_k^+\|n_k\rangle = \sqrt{(n_k+1)}\|n_k + 1\rangle$	$a_k\|n_k\rangle = \sqrt{(n_k+1)}\|n_k + 1\rangle$
$a_k^+ a_k\|n_k\rangle = n_k\|n_k\rangle$	$a_k a_k^+\|n_k\rangle = n_k\|n_k\rangle$
$a_k\|0\rangle = 0$	$a_k^+\|0\rangle = 0$

4. Applications

The analysis will, for illustration, first be applied to some well known nondispersive problems. The results are summarized in table 2.

Table 2.

Problem	Lagrangian	a_k and $q(t, z)$
Harmonic oscillator	$L(t, z) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}gx^2$	$a = \frac{1}{(2m\hbar\omega)^{1/2}}(m\omega Q + iP)$
	$\mathscr{L}(\omega) = m\omega^2 - g$	$x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a + a^+)$
Coupled oscillators	$L(t, z) = \sum_n \{\frac{1}{2}m\dot{x}_n^2 - \frac{1}{2}g(x_{n+1} - x_n)^2\}$	$a_k = \frac{1}{(2m\hbar\omega_k)^{1/2}}(m\omega_k Q_k + iP_k)$
	$\mathscr{L}_k(\omega_k, k) = m\omega_k^2 - g(1 - \cos kl)$	$x_n = \sum_k \left(\frac{\hbar}{2m\omega_k}\right)^{1/2}(a_k\, e^{iknl} + a_k^+\, e^{-iknl})$
Electromagnetic waves in vacuum	$L(t, z) = \frac{1}{2}\epsilon_0\left(\frac{\partial A}{\partial t}\right)^2 - \frac{1}{2\mu_0}\left(\frac{\partial A}{\partial z}\right)^2$	$a_k = \frac{1}{(2\epsilon_0\hbar\omega_k)^{1/2}}(\epsilon_0\omega_k A_k + iP_k)$
	$\mathscr{L}_k(\omega_k, k) = \epsilon_0\omega_k^2 - k^2/\mu_0$	$A = \sum_k \left(\frac{\hbar}{2\epsilon_0\omega_k}\right)^{1/2}(a_k\, e^{ikz} + a_k^+\, e^{-ikz})$

The next application will be longitudinal electromagnetic waves (plasmons) in a plasma system with an arbitrary number n of electron streams (velocities v_{0i}). With the scalar potential denoted ϕ and the electron oscillation amplitudes q_i we obtain

$$L(t, z) = \sum_i \left\{\frac{\epsilon_0}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 + N_i e_i \frac{\partial q_i}{\partial z}\phi + \frac{N_i m_i}{2}\left(\frac{\partial q_i}{\partial t} + v_{0i}\frac{\partial q_i}{\partial z}\right)^2\right\}. \tag{22}$$

Variation with respect to the variables we want to eliminate, that is q_i, yields relations between q_i and ϕ. We then obtain

$$\mathscr{L}_k(\omega_k, k) = \sum_{i=0}^{n} k^2\epsilon_0\left(1 - \frac{\omega_{pi}^2}{(\omega_k - v_{0i}k)^2}\right) = k^2\epsilon_L(\omega_k, k)$$

with $\omega_{pi}^2 = N_i e_i^2/m_i\epsilon_0$. The dispersion relation yields $2n$ solutions of positive and negative energy waves. The scalar potential is quantized according to

$$\phi(t, z) = \sum_k \sum_j \left(\frac{\hbar}{|k^2(\partial\epsilon_L/\partial\omega_k)_{\omega_{k,j}}|}\right)^{1/2}(a_{k,j}\, e^{ikz} + a_{k,j}^+\, e^{-ikz}). \tag{23}$$

We will now consider the nonlinear interaction between three quasimonochromatic waves as exemplified by plasmons. The nonlinear correction to the Lagrangian (22) is

$$L_{NL} = \sum_i \frac{N_i m_i}{2}\frac{\partial q_i}{\partial z}\left(\frac{\partial q_i}{\partial t} + v_{0i}\frac{\partial q_i}{\partial z}\right)^2$$

and we have $H_{NL} = -L_{NL}$. The earlier treatment is now generalized by assuming a slowly varying time dependence of the amplitudes besides the fast, linear variation

$\exp(-i\omega_k t)$. If we neglect correction terms including derivatives of the slowly varying amplitudes we can still use (19) and we can write the Hamiltonian in the general form (after simplification of notation)

$$H(t) = \sum_j \tfrac{1}{2} s_j \hbar\omega_j (a_j a_j^+ + a_j^+ a_j) + \hbar(\kappa a_1 a_2 a_3 + \kappa^* a_1^+ a_2^+ a_3^+). \tag{24}$$

We have only included interaction terms which are resonant when

$$\begin{aligned} \omega_1 + \omega_2 + \omega_3 &= 0 \\ k_1 + k_2 + k_3 &= 0. \end{aligned} \tag{25}$$

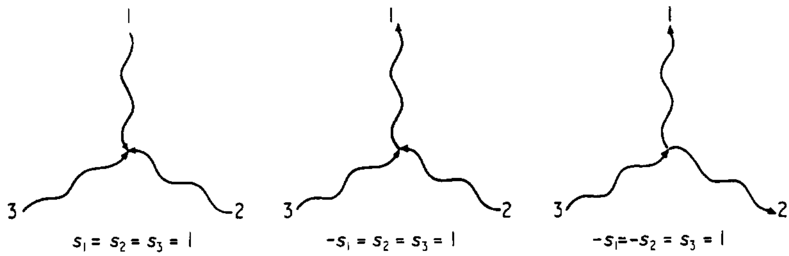

Figure 1. Three types of diagrams may illustrate the various processes which are associated with (25).

Heisenberg's equation of motion now yields

$$\frac{\partial a_1}{\partial t} = -i\omega_1 a_1 - i s_1 \kappa^* a_2^+ a_3^+. \tag{26}$$

The result for a_2 and a_3 is simply obtained by permutation of index. In the case of plasmons we obtain

$$\kappa = \sum_i \sqrt{\hbar \frac{N_i e_i^3}{m_i^2}} \times \frac{k_1(\omega_2 - v_{0i}k_2)(\omega_3 - v_{0i}k_3) + k_2(\omega_3 - v_{0i}k_3)(\omega_1 - v_{0i}k_1) + k_3(\omega_1 - v_{0i}k_1)(\omega_2 - v_{0i}k_2)}{\Pi_j(\omega_j - v_{0i}k_j)^2 \{|(\partial\epsilon_L/\partial\omega_k)_{\omega_{k,j}}\}^{1/2}}$$

which is in agreement with the nonlinear coefficient derived by Wilhelmsson (1969), where a normal-mode analysis of the differential equations is used. The result agrees as well with the special case $i = 1, 2$, treated by Fukai and Harris (1971). The complications in dealing with the canonical transformations even in the two stream case (Fukai and Harris 1971) are effectively avoided by the present method.

Although the amplitudes a_{kj} are linearly related to observables such as the electric field, higher order correction terms may be included in the analysis and will then result in higher order derivatives of a_j in the left hand side of (26) (cf Askne 1972). The result is that we obtain different types of interactions dependent on the signs of s_i. If $s_1 = s_2 = s_3 = 1$ we obtain explosive interactions (see for example Engelmann and Wilhelmsson 1969) while if $-s_1 = s_2 = s_3 = 1$ the interaction is an up-conversion process and $-s_1 = -s_2 = s_3 = 1$ is a down-conversion process. If we linearize by assuming that the a_3 field is so large that it may be assumed to be constant and described by a c number we obtain

$$\begin{aligned}\frac{\partial a_1}{\partial t} &= -\mathrm{i}\omega_1 a_1 - \mathrm{i}s_1\gamma^* \,\mathrm{e}^{\mathrm{i}\omega_3 t}\, a_2^+ \\ \frac{\partial a_2^+}{\partial t} &= +\mathrm{i}\omega_2 a_2^+ + \mathrm{i}s_2\gamma\, \mathrm{e}^{-\mathrm{i}\omega_3 t}\, a_1\end{aligned} \tag{27}$$

with solutions ($\gamma = \gamma^*$ for simplicity)

$$a_1 = \left\{a_{1,0}\cosh\surd(s_1s_3)\gamma t + \mathrm{i}\left(\frac{s_1}{s_3}\right)^{1/2} a_{2,0}^+ \sinh\surd(s_1s_3)\gamma t\right\}\mathrm{e}^{-\mathrm{i}\omega_1 t}$$

$$a_2^+ = \left\{a_{2,0}^+\cosh\surd(s_1s_3)\gamma t - \mathrm{i}\left(\frac{s_1}{s_3}\right)^{1/2} a_{1,0} \sinh\surd(s_1s_3)\gamma t\right\}\mathrm{e}^{+\mathrm{i}\omega_2 t}.$$

If we, like Louisell *et al* (1961), consider interaction between cavity modes due to a modulated dielectric medium: we have $s = \text{sign}\ \omega$ and with $\omega_1 + \omega_2 + \omega_3 = 0$; $-s_1 = s_2 = s_3 = 1$, that is $|\omega_1| = |\omega_2| + |\omega_3|$. For an exact knowledge of the number of input photons n_{10} and n_{20} we obtain

$$\begin{aligned} n_1(t) &= \langle a_1 a_1^+\rangle = n_{10}\cos^2\gamma t + n_{20}\sin^2\gamma t \\ n_2(t) &= \langle a_2^+ a_2\rangle = n_{20}\cos^2\gamma t + n_{10}\sin^2\gamma t. \end{aligned} \tag{28}$$

On the other hand if $-s_1 = -s_2 = s_3 = 1$, that is $|\omega_3| = |\omega_1| + |\omega_2|$, we obtain

$$\begin{aligned} n_1(t) &= \langle a_1 a_1^+\rangle = n_{10}\cosh^2\gamma t + (n_{20}+1)\sinh^2\gamma t \\ n_2(t) &= \langle a_2 a_2^+\rangle = n_{20}\cosh\gamma t + (n_{10}+1)\sinh^2\gamma t. \end{aligned} \tag{29}$$

In the first case we have a frequency conversion with $n_1 + n_3$ fixed while in the second case we have amplification with $n_1 + n_3$ increasing due to the pump wave a_3.

The final results are in agreement with Louisell *et al* (1961). Due to the above treatment we can, however, conclude that in any linear interaction we will find the same final results which means that active amplification can be stimulated by zero point noise. This is not the case for the passive conversion case.

Appendix

We consider a general quadratic or bilinear Lagrangian given by

$$L(t,z) = \frac{1}{2}\sum_{i,j}\left(a_{i,j}\frac{\partial q_i}{\partial z}\frac{\partial q_j}{\partial z} + b_{i,j}\frac{\partial q_i}{\partial t}\frac{\partial q_j}{\partial z} + c_{i,j}\frac{\partial q_i}{\partial t}\frac{\partial q_j}{\partial t} + d_{i,j}\frac{\partial q_i}{\partial z}q_j + e_{i,j}\frac{\partial q_i}{\partial t}q_j + f_{i,j}q_iq_j\right). \tag{A1}$$

Assuming

$$q_i(t,z) = \tfrac{1}{2}\sum_k (Q_{ik}\,\mathrm{e}^{\mathrm{i}kz} + Q_{ik}^+\,\mathrm{e}^{-\mathrm{i}kz}) \tag{A2}$$

with the time dependence of Q_{ik} given by $\exp(-\mathrm{i}\omega_k t)$ where $\omega_k = -\omega_{-k}$ we obtain

$$\int L\,\mathrm{d}z = \tfrac{1}{4}\sum_{i,j,k}(\mathscr{L}_{i,j,k}Q_{i,k}Q_{j,k}^+ + \mathscr{L}_{i,j,k}^*Q_{ik}^+Q_{jk}) \tag{A3}$$

with

$$\mathscr{L}_{i,j,k} = a_{i,j}k^2 - b_{i,j}\omega_k k + c_{i,j}\omega_k^2 + \mathrm{i}d_{i,j}k - \mathrm{i}e_{i,j}\omega_k + f_{i,j}. \tag{A4}$$

With $a_{i,j} = a_{j,i}, b_{i,j} = b_{j,i}, c_{i,j} = c_{j,i}, d_{i,j} = -d_{j,i}, e_{i,j} = -e_{j,i}$, and $f_{i,j} = f_{j,i}$ we obtain

$$\mathscr{L}_{i,j,k} = \mathscr{L}^*_{j,i,k} = \mathscr{L}_{j,i,-k} = \mathscr{L}^*_{i,j,-k} \tag{A5}$$

which is the case for loss-free media. Variation with respect to Q_{jk}^+ and Q_{jk} yields respectively

$$\sum_i \mathscr{L}_{i,j,k}Q_{ik} = 0 \qquad \sum_i \mathscr{L}^*_{i,j,k}Q_{ik}^+ = 0. \tag{A6}$$

From these relations we can express Q_{ik} in one of the amplitudes which we denote Q_k

$$Q_{ik} = \Lambda_{i,k}Q_k \tag{A7}$$

and we then obtain from (A6) the dispersion relation

$$\sum_i \mathscr{L}_{i,j,k}\Lambda_{i,k} = 0. \tag{A8}$$

We introduce $\mathscr{L}_k$ by

$$\mathscr{L}_k = \sum_{i,j} \mathscr{L}_{i,j,k}\Lambda_{i,k}\Lambda^*_{j,k} = \mathscr{L}^*_k \tag{A9}$$

and can then write (A3)

$$\int L\,\mathrm{d}z = \tfrac{1}{4}\sum_k \mathscr{L}_k(Q_kQ_k^+ + Q_k^+Q_k) \tag{A10}$$

in agreement with (6). The momentum is given by

$$p_i = \frac{\partial L}{\partial(\partial q_i/\partial t)} \tag{A11}$$

and the Hamiltonian by

$$H(t,z) = \sum_i p_i\frac{\partial q_i}{\partial t} - L(t,z). \tag{A12}$$

From (A1) we obtain after integration over unit length

$$\int H\,\mathrm{d}z = \tfrac{1}{4}\sum_{i,j,k} (\mathscr{H}_{i,j,k}Q_{ik}Q_{jk}^+ + \mathscr{H}^*_{i,j,k}Q_{ik}^+Q_{jk}) \tag{A13}$$

with

$$\mathscr{H}_{i,j,k} = -a_{i,j,k}k^2 + c_{i,j}\omega_k^2 - \mathrm{i}d_{i,j}k - f_{i,j} = \mathscr{H}^*_{j,i,k} \tag{A14}$$

and it is found that

$$\mathscr{H}_{i,j,k} = \omega_k\frac{\partial\mathscr{L}_{i,j,k}}{\partial\omega_k} - \mathscr{L}_{i,j,k}. \tag{A15}$$

Expressing Q_{ik} and Q_{jk} in terms of Q_k we obtain

$$\int H\,\mathrm{d}z = \tfrac{1}{4}\sum_k \mathscr{H}_k(Q_kQ_k^+ + Q_k^+Q_k) \tag{A16}$$

with

$$\mathscr{H}_k = \sum_{i,j} \mathscr{H}_{i,j,k} \Lambda_{i,k} \Lambda^*_{j,k}. \tag{A17}$$

We then find from (A9) (A8), and (A15) that

$$\begin{aligned} \omega_k \frac{\partial \mathscr{L}_k}{\partial \omega_k} - \mathscr{L}_k &= \sum_{i,j} \left(\omega_k \frac{\partial \mathscr{L}_{i,j,k}}{\partial \omega_k} - \mathscr{L}_{i,j,k} \right) \Lambda_{i,k} \Lambda^*_{j,k} \\ &\quad + \sum_{i,j} \omega_k \mathscr{L}_{i,j,k} \Lambda_{i,k} \frac{\partial \Lambda^*_{j,k}}{\partial \omega_k} + \sum_{i,j} \omega_k \mathscr{L}_{i,j,k} \Lambda^*_{j,k} \frac{\partial \Lambda_{i,k}}{\partial \omega_k} \\ &= \sum_{i,j} (\omega_k \frac{\partial \mathscr{L}_{i,j,k}}{\partial \omega_k} - \mathscr{L}_{i,j,k}) \Lambda_{i,k} \Lambda^*_{j,k} = \mathscr{H}_k \end{aligned} \tag{A18}$$

in agreement with (9). It should be noted that ω_k and k are considered as independent variables.

With the symmetries assumed we obtain $\mathscr{L}_k = \mathscr{L}_{-k}$ and $\mathscr{H}_k = \mathscr{H}_{-k}$. From (A10) and (A16) and with $Q_k = Q^{\pm}_{-k}$ we realise that we obtain the same result for summation over positive k as over negative k.

Although the derivation starts from an averaged Lagrangian quantity the important quantity may as well be obtained directly from the differential equations if we include an external source term for normalization of $\mathscr{L}_k(\omega_k, k)$(cf Askne 1972).

References

Alekseev A I and Nikitin Yu P 1966 *Sov. Phys.–JETP* **23** 608–14

Askne J 1972 *Int. J. Electron.* to be published

Engelmann F and Wilhelmsson H 1969 *Z. Naturforsch.* **24a** 206–7

Fukai J and Harris E G 1971 *Phys. Fluids* **14** 1748–52

Harris E G 1970 *Advances in Plasma Physics* ed A Simon and W B Thomson (New York : Wiley) pp 156–248

Hopfield J J 1958 *Phys. Rev.* **112** 1555–67

Jauch J M and Watson K M 1948 *Phys. Rev.* **74** 950–7

Kong J A 1970 *J. appl. Phys.* **41** 554–9

Louisell W H, Yariv A and Siegman A E 1961 *Phys. Rev.* **124** 1646–54

Musha T 1964 *J. appl. Phys.* **35** 3273–9

Sturrock P A 1960 *J. appl. Phys.* **31** 2052–6

Wilhelmsson H 1969 *J. Plasma Phys.* **3** 215–26